Int. J. Solids Structures, 1970, Vol. 6, pp. 423 to 432. Pergamon Press. Printed in Great Britain

THE APPLICATION OF POINCARÉ'S SPHERE TO PHOTOELASTICITY

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Abstract—Polarimetry problems can be dealt with simply with the help of Poincaré's representation [1]; as this method is not widely appreciated its properties are developed in Section 1. Thereafter, in Section 2, an automatic method for measuring polarization forms is described.

Finally, Section 3 describes the application of the preceding results to both automatic two- and three-dimensional photoelasticity by scattered light.

1. PROPERTIES OF POINCARÉ'S SPHERE

1.1 *Poincaré's sphere*

IN ORDER to avoid any ambiguity, space will be described in relation to three orthogonal planes, the axis OZ being parallel to and having the same sense as the direction of propagation of the light.

A right-handed rotation about an oriented axis when facing in the direction of propagation is defined as positive.

Consider an elliptically polarized beam of light of semi-axes a and b (Fig. 1), whose major axis makes an angle α, defined to the nearest π, with the reference axis OX. The direction of rotation of the electric vector on the ellipse is positive or negative according to whether the light is right- or left-handed.

Poincaré's sphere may be defined, with reference to Fig. 2, as follows: its radius is unity; its centre is at O; the axis OZ meets it at P' and P respectively with coordinates -1 and $+1$, and the equatorial plane is defined by the plane XOY.

An elliptical vibration is defined on Poincaré's sphere as follows:

On the equator the point m is found such that,

$$\text{angle } O\hat{X}m = 2\alpha$$

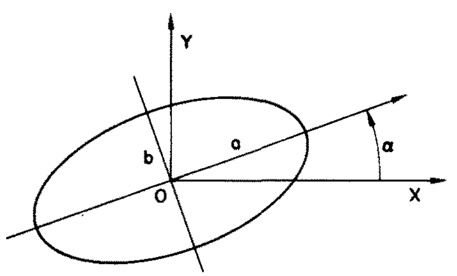

FIG. 1.

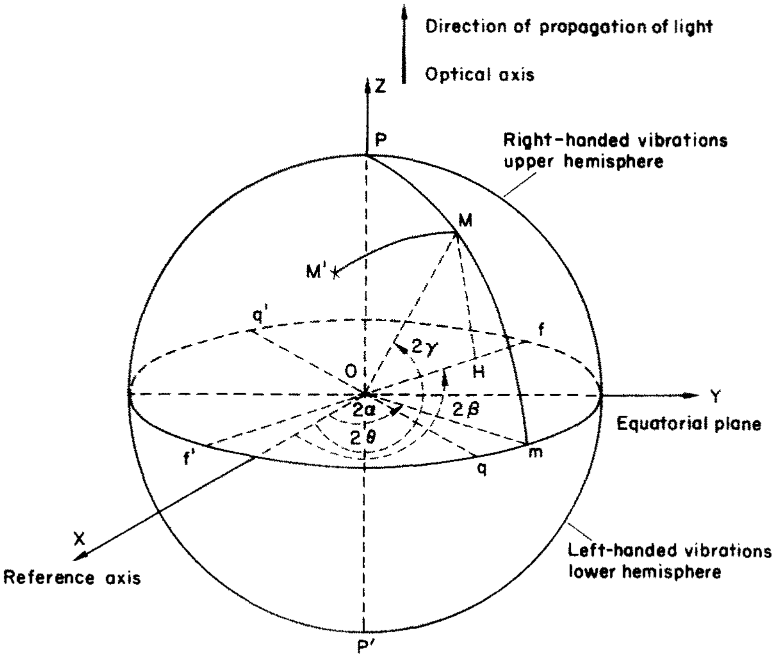

FIG. 2. Poincaré's sphere.

that is, twice the azimuth, the angle between OX and the major axis of the ellipse. The line of longitude $P'mP$ is then drawn. Next, a line is drawn from O to meet $P'mP$ at M such that,

$$\text{angle } M\hat{O}M = 2\gamma$$

where

$$\tan \gamma = b/a \leq 1.$$

2γ is considered positive in the upper hemisphere and the final point M represents a right-handed vibration.

2γ is considered negative in the lower hemisphere and the final point represents a left-handed vibration.

Since $2\gamma \leq \pi/2$ there is a one-to-one correspondence between a point on the sphere and an elliptic vibration.

The lines of latitude represent vibrations of the same ellipticity and of the same handedness; the lines of longitude correspond to vibrations which have the same inclination of the elliptic axes (azimuths); the equator corresponds to states of linear vibration; the poles P and P' correspond to right- and left-circular vibrations.

1.2 *The representation of a birefringent plate*

A birefringent plate has two orthogonal axes of propagation, the difference in speeds of propagation along the fast and slow axes corresponds to a difference in phase ϕ.

On Poincaré's sphere a birefringent plate is represented by a rotation through an angle ϕ in a positive direction about an axis $q'q$ situated in the equatorial plane, points q and q' representing twice the angles between OX and the fast and slow axes respectively (Fig. 2).

Thus, a vibration represented by M as it enters the birefringent plate is represented on exit by point M', obtained by rotating M along an arc through the angle ϕ in a positive direction about the axis $q'q$.

1.3 *The representation of a rotatory power*

A rotatory power of value R is represented on Poincaré's sphere by a rotation through and angle $2R$ about the axis OZ in a positive or negative direction according to whether the rotatory power is right- or left-handed. It modifies neither the ellipticity nor the handedness of the ellipse.

1.4 *The action of a series of birefringent plates on a vibration*

A series of birefringent plates whose axes are displaced is represented on Poincaré's sphere by a succession of rotations, the order being that in which the light meets the different birefringent plates, this being noncommutative.

It can be proved that this succession of rotations about axes situated in the equatorial plane is equivalent to a single rotation about an axis situated in the same plane accompanied by a rotation about OZ, whence the theorem first stated by Poincaré:

A series of birefringent plates is equivalent to a single birefringent plate accompanied by a rotatory power.

1.5 *Representation of a polarizer or linear filter*

Consider a linear filter whose axis of polarization forms an angle β with the reference direction. In the equatorial plane of Poincaré's sphere (Fig. 2) two points f and f' are placed such that

$$\text{angle } O\hat{X}f = 2\beta$$

and

$$\text{angle } O\hat{X}f' = 2\beta + \pi.$$

A vibration represented by M with energy E_0 is represented by f after passing through the filter, its energy E being defined by:

$$E = TE_0 \frac{f'H}{f'f}.$$

where H is the foot of the orthogonal projection of M onto $f'f$ and T is the transmission factor of the filter.

1.6 *These properties having been considered*

All the principles of measurement of the apparatus of classical polarimetry are now clear. However, in classical polarimetry, the problems of measurement can always be reduced to the determination of an extinction and this, for the sake of simplicity, leads one to work only in whole numbers of fringes; that is, in a discontinuous manner. It will now be shown that by a new method of analysing polarization forms, it is possible to work in a continuous manner.

2. THE ANALYSIS OF POLARIZATION FORMS OF LIGHT

An elliptically polarized light represented by a point on Poincaré's sphere is characterized by its ellipticity $\tau = b/a$, the orientation of its semi-major axis a and its direction.

It will be shown that the measurement of the ellipticity and of the orientation of the semi-major axis, can be effected by a simple energy measurement, with automatic display of these two parameters.

2.1. *Theory*

Consider an elliptically polarized light beam whose major axis forms an angle α with the horizontal axis OX (Fig. 3), having semi-major and semi-minor axes a and b respectively. Let $\tau = b/a = \tan \gamma$.

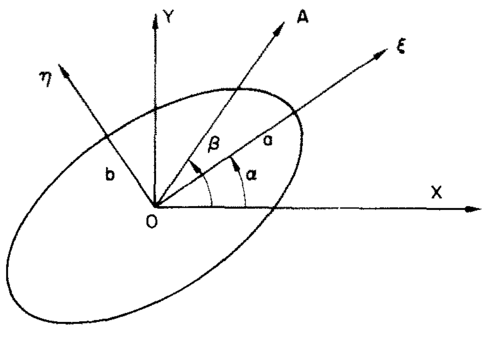

FIG. 3.

The light energy E transmitted by an analyser with an axis of polarization OA, forming an angle β with OX can be expressed (to the nearest transmission factor) as

$$E = \frac{a^2 + b^2}{2}\left[1 + \frac{1 - \tau^2}{1 + \tau^2}\cos 2(\beta - \alpha)\right]$$

$$= \frac{a^2 + b^2}{2}[1 + \cos 2\gamma \cos 2(\beta - \alpha)].$$

The analyser may be rotated with a constant frequency ω. The light-flux is then resolved into two components E_1 and E_2, one constant and the other alternating.

$$E_1 = \frac{a^2 + b^2}{2}$$

$$E_2 = \frac{a^2 + b^2}{2}\left[\frac{1 - \tau^2}{1 + \tau^2}\cos 2(\omega t - \alpha)\right]$$

If the light flux, consisting of these two components, is then allowed to fall upon a photomultiplier this will give, as its output, a modulated current containing the information of interest, the frequency of modulation being double that of rotation of the analyser.

By using a suitable load impedance the current modulation may be transformed into a voltage. Thus two voltages V_1 and V_2 are obtained, proportional to E_1 and E_2 respectively.

The ratio $(1 - \tau^2)/(1 + \tau^2) = \cos 2\gamma$ of the amplitude of the alternating voltage V_2 to that of the d.c. voltage V_1 defines ellipticity. The phase 2α of the alternating signal relative to a reference signal $\cos 2\omega t$, which vary at twice the frequency of rotation of the analyser, defines the position of the semi-major axis.

Thus, on Poincaré's sphere (Fig. 4), the vibration is represented by point M or M' according to whether it is right- or left-handed, following the conventions adopted.

If a quarter wave plate is placed in the light path before the rotating analyser and arranged so that its fast axis is coincident with the major axis of the ellipse (which has already been determined) then the phase of the alternating signal will indicate whether the vibration is represented by M or M'.

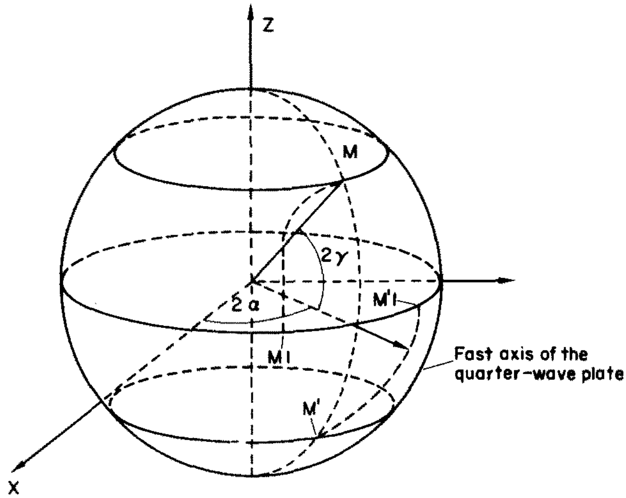

FIG. 4.

2.2. *Practical application*

The electronic apparatus is shown schematically in Fig. 5. Because information is localized in the zero frequency term on the one hand and in the double frequency of the rotation of the analyzer on the other, filters may be used in such a way as to eliminate the

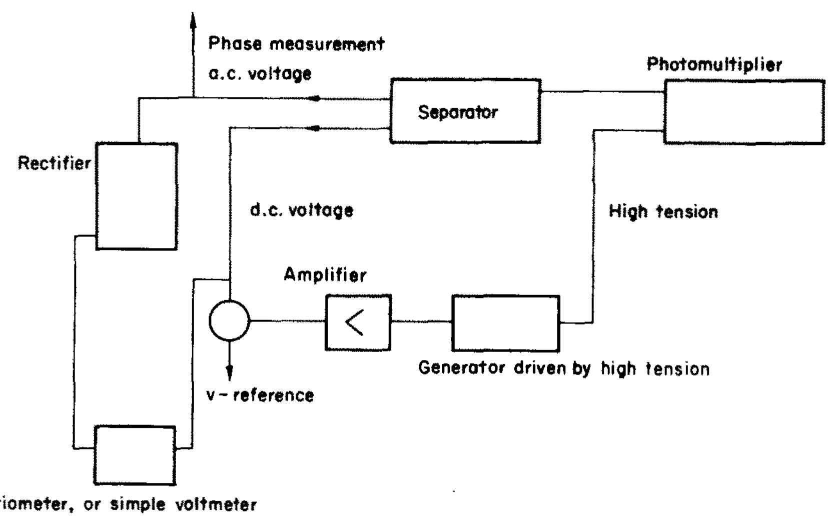

FIG. 5.

effects of "noise" in the photomultiplier. Finally, by comparing the d.c. voltage with a fixed, reference voltage the gain of the photomultiplier circuit may be controlled so as to maintain the d.c. voltage at a steady value. This allows elimination of the effects of fluctuation of the energy output of the light source. With the light linearly polarized the ratio of the rectified alternating voltage to the d.c. voltage is set to unity by means of a potentiometer. A ratiometer or, possibly, a simple voltmeter, can then be used to determine the quantity $(1 - \tau^2)/(1 + \tau^2)$.

It should be observed that this method of measurement remains valid even if there is an incoherent luminous background on condition however, that this latter stays constant and that the light form being studied can be varied up to the linearly polarized form, which is the case in both plane and three-dimensional photoelasticity, since all that is then required is to introduce a variable phase-transformer whose axis is coincident with the previously determined semi-major axis.

3. APPLICATIONS

Every transparent body is an *operator* on the polarized wave train and the methods which have been described can readily be applied to the determination of the coefficients of this operator; all that is necessary is that the operator should vary the polarization form of the light falling upon the body under examination. However this discussion will be limited to applications to photoelasticity.

3.1 *Theory and description of an automatic two-dimensional photoelasticimeter*

Suppose that a beam of monochromatic light, say, right circularly polarized, be directed to a point on the model. Further, suppose that at this point the difference in phase ϕ is less than $\pi/2$, and that the fast axis forms an angle θ with the reference direction as it leaves the model. According to Poincaré's representation the vibration will be represented by the point M, obtained by rotating P through an angle ϕ about axis $q'q$ (Fig. 6).

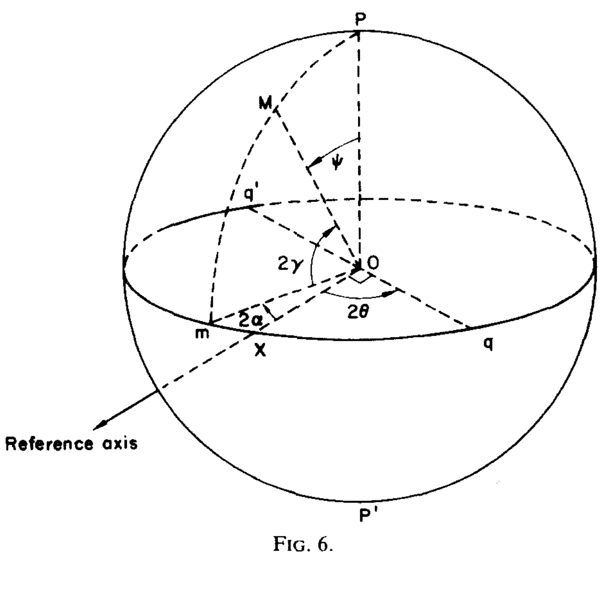

FIG. 6.

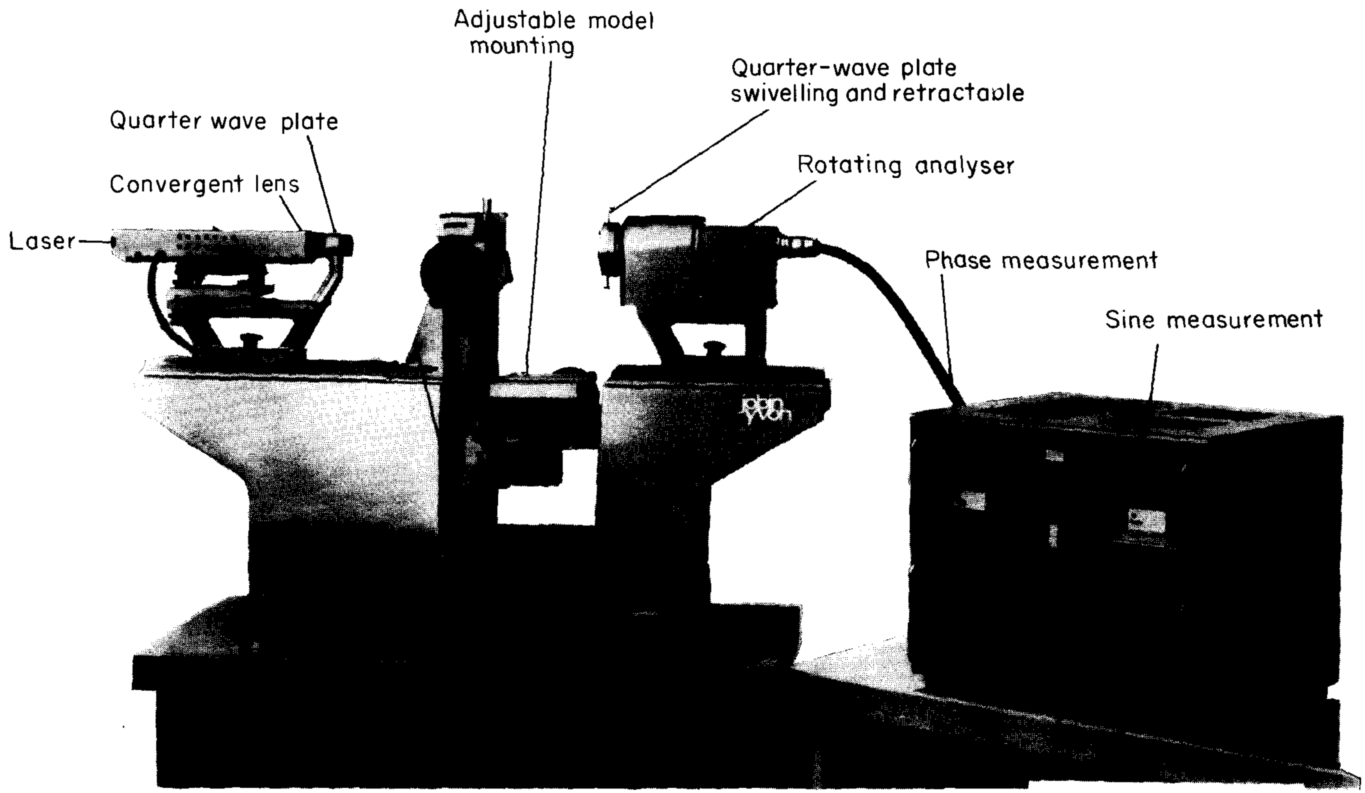

FIG. 7. Two dimensional photoelasticimeter.

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FIG. 10

By the method previously described for analysing polarization forms, the value of twice the angle α between the major axis of the ellipse and the reference direction and the value 2γ can be obtained automatically; note that $\cos 2\gamma$ is equal to the sine of the phase-difference, and 2θ equals $2\alpha + \pi/2$.

The prototype of the apparatus is shown in Fig. 7. The light-source is a laser; a quarter-wave plate converts this to circularly polarized light and a convergent lens provides a very small spot of light on the model. The model may be moved in two orthogonal directions.

On leaving the model the beam of light passes through the rotating analyser and then falls upon the photocathode of the photomultiplier. The values of $\sin \phi$ and of θ are read from the two digital voltmeters.

A swivelling, retractable quarter-wave plate permits, on the one hand, in the absence of the model, the calibration of $\sin \phi$, and, on the other hand, the model being in position, allows determination of whether the vibration is right- or left-handed. It is, however, advantageous to restrict the difference in phase to below $\pi/2$. This leads to the use of materials which are not very photoelastic, such as Plexiglass.

The accuracy of measurement is of the order of a thousandth of a radian for ϕ and about a degree in the orientation of the fast axis.

Depending on the sign of Brewster's constant the algebraically greatest stress is either orthogonal to or coincident with the fast axis.

3.2 *Theory and description of a three-dimensional scattered light photoelasticimeter*

The photoelastic model is a transparent body, subject to mechanical loading, whose birefringence varies from point to point. Suppose that (Fig. 8) on such a body, immersed in a liquid of the same refractive index as itself, there falls a beam of SM of *non-polarized* light; a point such as M becomes visible due to light scattered transversely to the beam SM. This scattered light has properties which are different from those of the incident light; thus at M, light emitted along the orthogonal beam MO is *polarized* in the direction P, perpendicular to plane SMO.

However, between point M and point N where it leaves the body, beam MO passes through a series of birefringent plate elements. The sum of these, as the classical representation of Poincaré's sphere clearly shows, is equivalent to the effect of a single birefringent plate accompanied by a rotatory power: therefore, at N, the light has become *elliptically* polarized.

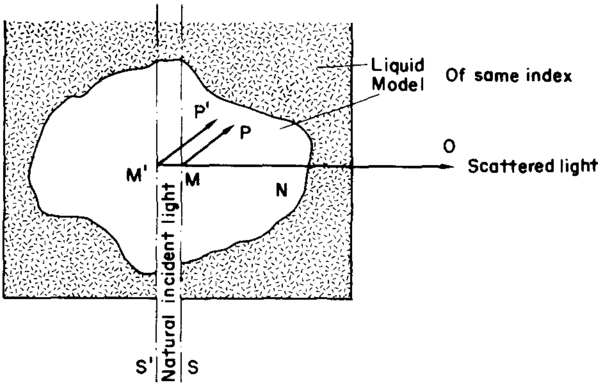

FIG. 8. Theory of three-dimensional photoelasticity by scattered light.

If the incident beam were $S'M'$ (M' being on the straight line MO) light scattered at M' along $M'O$ would still be polarized in the direction perpendicular to the plane of $S'M'O$, that is, it would have the same characteristics as before at M; but, due to the addition of the elementary birefringent $M'M$, the elliptically polarized light beam arriving at N would be different. *Now, the study of the characteristics of these two light beams allows the characteristics of birefringent plates equivalent to paths MN and M'N to be found; the comparison of these characteristics of the equivalent birefringent plates gives those of the elementary birefringent plate M'M. This is the theory of the proposed new method for three-dimensional photoelasticity.*

Since this theory depends upon the determination of differences it can only be used fruitfully if *a very accurate method is available for measuring the characteristics of the emergent elliptic light*, that is, ellipticity b/a, orientation of the semi-major axis a and direction. The method of analysing polarization forms previously described is suitable.

This method can be applied to the determination of the characteristics of the birefringent plate equivalent to a path such as MN in the body (Fig. 8). All that is necessary is to rotate the body about axis MN.

This being so, the direction of polarization of light at M, fixed in space, rotates in relation to the axes of the equivalent birefringent plate and this modifies the characteristics of elliptic light vibration obtained at N: it can then be shown that two vibrations which, linear at M, give at N elliptic vibrations of the same ellipticity b/a, are necessarily symmetrical in relation to the axes of the equivalent birefringent plate or to the bisectors of those axes, whatever the rotatory power accompanying the equivalent birefringent plate. Moreover, if the direction of polarization is made to coincide with an axis of the birefringent plate, the ratio between the alternating voltage is maximal, whereas it is minimal if the direction of polarization is made to coincide with a bisector of the axes: all ambiguity in orientation is thus eliminated. Finally, by dividing the minimum ratio by the maximum, one obtains the cosine of the difference in phase of the equivalent birefringent plate, which permits the determination of this difference in phase, and the measurement of the phase of the alternating signal provides knowledge of the rotatory power.

In the application shown diagrammatically in Fig. 9 the illuminated point in the model which is to be examined is held fixed. The model can be translated in three orthogonal directions; also, it can be rotated about the axis along which the scattered light is observed.

The lighting unit consists of a lamp (1), having a constant current supplied to it, a condenser (2) and a pin-hole (3) which acts as source. The beam coming from this source enters the body and the image of that part of this beam which surrounds point (4) is formed, by a fixed system of lenses, on a slit of adjustable width (5), at right-angles to the axis of the beam of light in the model. Thus, a beam of scattered light is isolated, emitted from the point studied along the axis coincident with the axis of observation and rotation.

The beam of light then passes through the rotating analyser (6), and then through a filter (7), and falls on the photomultiplier (8). A swivelling quarter-wave-plate can be introduced directly in front of the slit (9).

Experiments performed with the industrial bench, illustrated in Fig. 10, have shown that the accuracy obtained was of the order of a thousandth of a fringe for a phase-difference in the equivalent birefringent plate, of the order of a tenth of a degree in the position of the axes of the latter, and of a degree for the rotatory power. An accurate determination of the birefringences within a body can thus be obtained.

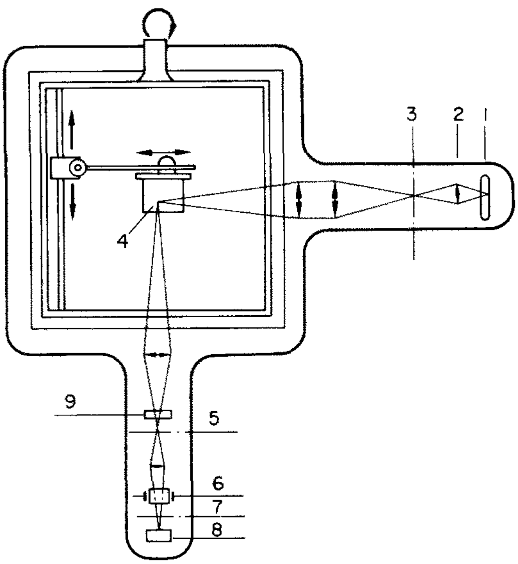

FIG. 9. Layout of the three-dimensional photoelastic bench using scattered light.

The practical experience of the next few months will show to what extent the fundamental advantages of this method, in particular its universality, the integrity it assures models and the flexibility it allows in the choice of materials used to make these models, will find favour with those who use it and will help to extend the use of three-dimensional photoelasticity by scattered light.

CONCLUSIONS

It has been shown that Poincaré's representation is readily adapted to the problems of polarimetry and photoelasticity and that the automatic method of analysing polarization forms of light can be easily understood from this representation. It is felt that these very precise methods of ellipsometry will find many other future applications.

The author wishes to thank his teacher, M. de Leiris, General Engineer of the Corps of Naval Constructors, M. Ballet, Chief Engineer of the Corps of Naval Constructors and Head of the Group Laboratories, and the "Jobin et Yvon" Company who, having built the industrial prototype of the three-dimensional photoelasticimeter, are at present constructing the two-dimensional automatic bench.

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(*Received* 25 *March* 1969)

Абстракт—Задачи поляриметрии решаются с помощью метода Пуанаре. Так как этот метод до сех пор широко не применяется, поэтому в первой части работы принодится его свойства. Далее, во второй части дается описание автоматического метода измерения форм поляризации. Наконец, в третьей части работы, описывается применение выше полученных результатов для задач двух- и трех-мерной фотоупругости ирц рассечнном сьеме.